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SHORTER COMMUNICATION

HEAT TRANSFER IN LAMINAR FLOW OF HEAT-GENERATING FLUIDS IN A PARALLEL PLATE CHANNEL

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NOMENCLATURE

a,	thermal diffusivity;
A_i,	constant;
B,	$= Ud/\lambda$, Biot number;
d,	half of plate-distance;
H,	dimensionless heat flux density at the walls (13);
Pe,	$= u_m d/a$, Péclet number;
q,	heat flux density at the walls;
Q,	volumetric heat-generation rate;
Q_0,	specific volumetric heat-generation rate;
t,	time;
T,	fluid temperature;
T_0,	inlet and initial fluid temperature;
T_F,	temperature of the fluid outside the walls;
u,	fluid velocity;
u_m,	mean fluid velocity;
U,	heat-transfer coefficient;
x,	longitudinal co-ordinate;
z,	transverse co-ordinate.

Greek symbols

$\alpha_i, \beta_i, \gamma_i$,	constants;
η,	$= (2/3Pe)(x/d)$, dimensionless longitudina co-ordinate;
Θ,	$= (T - T_0)/(T_F - T_0)$, dimensionless fluid temperatures;
λ,	heat conductivity;
Λ,	$= Q_0 d^2/\lambda$, dimensionless specific volumetric heat generation rate;
ξ,	$= z/d$, dimensionless transverse co-ordinate;
τ,	dimensionless time;
Ω,	dimensionless fluid temperature with null heat generation.

INTRODUCTION

IN PREVIOUS papers [1–4] we considered some problems related to the laminar flow of fluids with a volumetric rate of heat generation linearly dependent on the local temperature.

In paper [1] steady-state temperature profiles with parabolic and piston flow in circular tubes and fixed wall temperature are given. In [3] the problem is extended to non-Newtonian fluids following the power law; heat transfer is also considered. Paper [2] examines the unsteady state in circular tubes with piston flow when, starting from an arbitrary temperature distribution, the temperature at which the heat-generation rate becomes null, is established in the inlet section and on the wall of the duct. Lastly, paper [4] considers a countercurrent heat exchanger with heat generation in the inner fluid; temperatures are considered only axially variable, the other usual assumptions also being made.

In this communication the previous researches are extended to consider the transient heat transfer in fluids flowing laminarly in a parallel plate channel. The transient conditions are determined, starting from isothermal conditions with a null heat-generation rate, and giving a step change in the wall temperature or in the temperature of the fluid outside the walls.

The analogous problem for fluids flowing in circular tubes is developed by di Federico [5].*

MATHEMATICAL TREATMENT

As in the previous papers the following assumptions are made: Fourier's law is valid; the physical properties of the fluid are constant; the electromagnetic, nuclear and radioactive energies, the viscous dissipations and the axial conduction of heat are negligible; the heat-generation rate depends linearly on the local temperature:

$$Q = Q_0(T - T_0). \tag{1}$$

The considered motion is the fully developed laminar flow of a Newtonian fluid between the rigid planes $z = \pm d$. Let x be the co-ordinate in the direction of motion, while none of the quantities are dependent on the y co-ordinate; thus the velocity is given by:

$$u = \frac{3}{2} u_m \left[1 - \left(\frac{z}{d}\right)^2\right]. \tag{2}$$

* The references on heat transfer in heat generating fluids are discussed in a previous paper [3]. Recently Sparrow, Novotny and Lin [6] have studied the heat transfer in laminar flow in a parallel-plate channel for steady state conditions and heat-generation rate depending on space variables, while the analogous problem in the entrance region of a parallel-plate channel is analysed by Novotny and Eckert [7].

Introducing dimensionless variables, the energy equation is:

$$\frac{\partial\Theta}{\partial\tau} + (1 - \xi^2)\frac{\partial\Theta}{\partial\eta} = \frac{\partial^2\Theta}{\partial\xi^2} + \Lambda\Theta. \tag{3}$$

The following boundary conditions are considered:

$$\Theta(0, \eta, \xi) = \Theta(\tau, 0, \xi) = 0 \tag{4}$$

$$B[1 - \Theta(\tau, \eta, \pm 1)] = \left(\frac{\partial\Theta}{\partial\xi}\right)_{\xi = \pm 1} \text{ for } \tau > 0. \tag{5}$$

To state condition (5), a constant heat-transfer coefficient U is assumed to give the heat transfer from the walls to the fluid outside the channel, also with unsteady conditions.

Condition (5) represents a step in the temperature of the fluid outside the walls of the channel; when $B = \infty$, it corresponds to a step in the temperature of the walls.

From paper [8] we see that the solution of equation (3) with boundary conditions (4) and (5) may be obtained from the solution for the case without heat generation having the same boundary conditions stating:

$$\Theta = \Omega \exp[\Lambda\tau] - \Lambda \int_0^\tau \Lambda \exp[\Lambda\tau]\, d\tau \tag{6}$$

where Ω is the solution of equation (3) when $\Lambda = 0$ and with boundary conditions (4) and (5).

The solution is now developed considering the approximate temperature distribution that may be obtained from the steady state distribution (still with $\Lambda = 0$) following the method developed by Siegel [9] for $B = \infty$. With this method the solution is obtained as a series expansion about the steady-state condition and with the approximation that the transient solution is only required to satisfy an integrated form of the energy equation, while it will exactly satisfy the equation in differential form at very large times. The form of the solution is:

$$\Omega = 1 - \sum_i^{0\ldots n-1} A_i\varphi_i(\xi) \exp[-a_i\eta] - \sum_i^{n\ldots\infty} A_i\varphi_i(\xi) \exp[-\beta_i\tau] \tag{7}$$

where n is such that:

$$\gamma_{n-1}\,\eta < \tau \leqslant \gamma_n\eta \tag{8}$$

with:

$$\gamma_n = a_n/\beta_n \tag{9}$$

The constants A_i, a_i and the functions φ_i are those of steady state solution, while the constants β_i are given by the relation:

$$\beta_i = -\frac{\left(\dfrac{d\varphi_i}{d\xi}\right)_1}{\int_0^1 \xi\varphi_i\, d\xi} \tag{10}$$

For the case $B = \infty$ the development of Siegel [9] is considered. For the case $B \neq \infty$ we apply Siegel's method to the steady-state solution given by van der Does de Bye and Schenk [10].

Introducing equation (7) in (6) and considering in the integration that n is a discontinuous function of τ, one obtains:

$$\left.\begin{aligned}\Theta = 1 &+ \sum_i^{0\ldots 4}\frac{\Lambda}{\beta_i - \Lambda}A_i\varphi_i - \sum_i^{0\ldots n-1}\frac{\beta_i}{\beta_i - \Lambda}A_i\varphi_i \\ &\exp[(\Lambda - \beta_i)\gamma_i\eta] - \sum_i^{n\ldots\infty}\frac{\beta_i}{\beta_i - \Lambda}A_i\varphi_i \\ &\exp[(\Lambda - \beta_i)\tau]\end{aligned}\right\} \tag{11}$$

with n defined by relation (8).

From the previous equation it is obvious that *only for $\Lambda < \beta_0$ is it possible to get a steady state.* The steady-state solution is given by equation (11) with $n = \infty$.

When $B = \infty$, considering the five-term approximation given by Siegel, it is:

$$\left.\begin{aligned}\Theta = 1 &+ \sum_i^{0\ldots 4}\frac{\Lambda}{\beta_i - \Lambda}\sum_j^{0\ldots 4} b_{ij}\cos\frac{(2j+1)\pi\xi}{2} \\ &- \sum_i^{0\ldots n-1}\frac{\beta_i}{\beta_i - \Lambda}\exp[(\Lambda - \beta_i)\gamma_i\eta]\,.\sum_j^{0\ldots 4} b_{ij} \\ &\cos\frac{(2j+1)\pi\xi}{2} - \sum_i^{n\ldots 4}\frac{\beta_i}{\beta_i - \Lambda} \\ &\exp[(\Lambda - \beta_i)\tau]\sum_j^{1\ldots 4} b_{ij}\cos\frac{(2j+1)\pi\xi}{2}\end{aligned}\right\} \tag{12}$$

Having thus obtained the temperature distribution, it is easy to calculate the heat transferred at the walls. The dimensionless parameter:

$$H \equiv \frac{qd}{\lambda(T_F - T_0)} = \left(\frac{\partial\Theta}{\partial\xi}\right)_{\xi = 1} \tag{13}$$

is considered. For $B \neq \infty$, H may be calculated introducing equation (11) in equation (5). For $B = \infty$, instead, H is calculated applying Fourier's law. Thus, from Siegel's five-term approximation, we have:

$$\left.\begin{aligned}H = \frac{\pi}{2}\Bigg[&\sum_i^{0\ldots 4}\frac{\Lambda}{\beta_i - \Lambda}\sum_j^{0\ldots 4} b_{ij}(2j+1)j^{-1} \\ &- \sum_i^{0\ldots n-1}\frac{\beta_i}{\beta_i - \Lambda}\exp[(\Lambda - \beta_i)\gamma_i\eta] \\ &\sum_j^{0\ldots 4} b_{ij}(2j+1)j^{-1} \\ &- \sum_i^{n\ldots 4}\frac{\beta_i}{\beta_i - \Lambda}\exp[(\Lambda - \beta_i)\tau] \\ &\sum_j^{1\ldots 4} b_{ij}(2j+1)j^{-1}\Bigg]\end{aligned}\right\} \tag{14}$$

Table 1

i	α_i $B = 1$	α_i $B = 10$	β_i $B = 1$	β_i $B = 10$	γ_i $B = 1$	γ_i $B = 10$	A_i $B = 1$	A_i $B = 10$
0	1·0000	2·4081	0·738	1·992	1·355	1·209	1·088	1·183
1	21·6802	29·1341	2·549	8·099	8·507	3·598	−0·117	−0·266
2	73·3044	86·5365	5·195	16·903	14·111	5·120	0·038	0·132

NUMERICAL EXAMPLES

For the five-term approximation relative to the case of fixed-wall temperature ($B = \infty$) the necessary numerical values are given in Siegel's paper [9]. Next, cases $B = 1$ and $B = 10$ are considered, taking the numerical values for a three-term approximation of the steady-state solutions from [10]. The values of β_i and γ_i calculated from (10) and (9) are given in Table 1 where the values of α_i and A_i are also reported.

The obtained solutions are such that, for $0 \leqslant \tau \leqslant \gamma_0\eta$, H is a function of τ only: this function is given in Fig. 1 for cases $\Lambda = 0$ and $B = \infty$; 10; 1. Case $\Lambda = 1$ is considered in Fig. 2 for $B = \infty$ and $B = 10$ (when $B = 1$ is $\beta_0 < \Lambda$). The heat is first transferred by the walls to the fluid flowing in the channel and, because of the resistance $1/B$, the heat flux is smaller for $B = 10$. Successively the heat transfer diminishes, but this diminution is larger for $B = \infty$, because of the rise in the fluid temperature due to the larger heat flux from the surroundings to the fluid and the consequent greater heat generation. Then the heat transfer inverts, going from the fluid to the walls. At high time values the heat transferred to the walls is greater for $B = \infty$, but this is not a general case. Indeed from the same Fig. 2, where the case of $\Lambda = 0{\cdot}25$, $B = \infty$ and $B = 1$ is also considered, we remark that at high time values the heat transferred at the walls is larger for $B = 1$. This may be explained considering that, for $B = 1$, the smaller heat transfer to the walls determined a rise in the fluid temperature and,

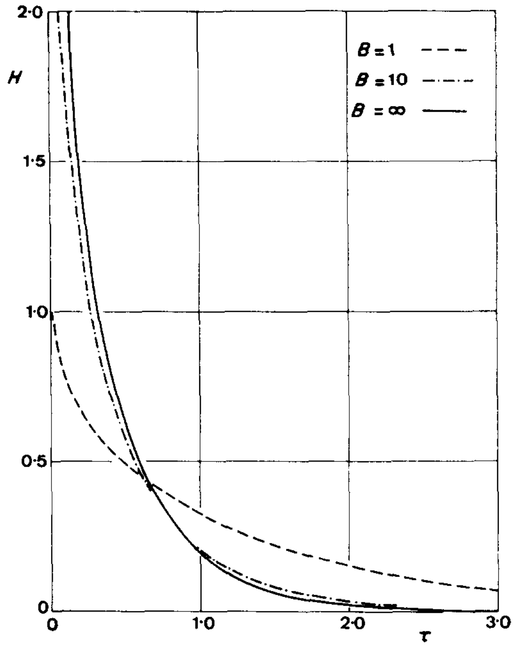

FIG. 1. Heat flux density at the walls vs. time ($0 \leqslant \tau \leqslant \gamma_0\eta$) for $\Lambda = 0$.

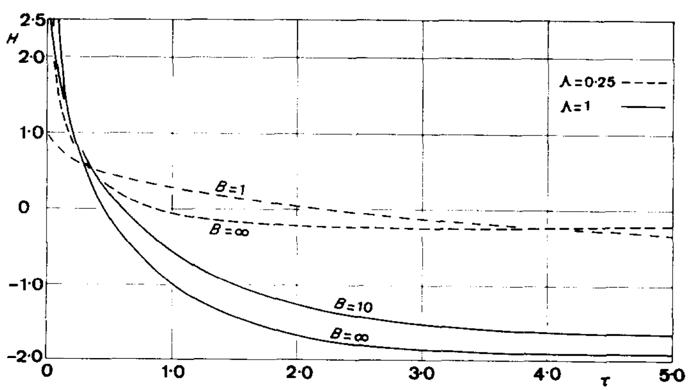

FIG. 2. Heat flux density at the walls vs. time ($0 \leqslant \tau \leqslant \gamma_0\eta$) for $\Lambda = 0{\cdot}25$ and $\Lambda = 1$.

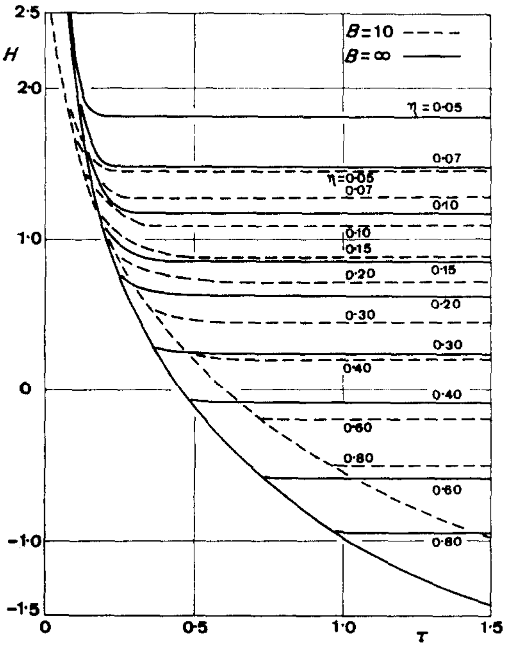

FIG. 3. Heat flux density at the walls vs. time at given values of longitudinal co-ordinate for $\Lambda = 1$.

consequently, an increase in heat generation, until the heat transfer also becomes greater than for $B = \infty$. Obviously the establishment at high time values of one or the other situation depends on the values of B.

Figure 3 gives the dimensionless heat flux H for $\Lambda = 1$ and, respectively, $B = \infty$ and $B = 10$ at several axial co-ordinates η. We remark that, at a given η, the heat transfer is no more time dependent after a finite time; this is due to the finite number of terms considered in calculating the summations, even if the time-dependent terms neglected give practically negligible contributions. The small number of terms considered makes a poor approximation of the results obtained at the smallest τ and η, also because the series we must consider in order to determine heat transfer do not have terms with alternate signs.

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